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LETTER TO THE EDITOR

Periodic orbit expansions for classical smooth flows

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Abstract. We derive a generalized Selberg-type zeta function for a smooth deterministic flow which relates the spectrum of an evolution operator to the periodic orbits of the flow. This relation is a classical analogue of the quantum trace formulae and Selberg-type zeta functions.

It is a characteristic feature of deterministic chaotic systems that despite their ergodic behaviour at asymptotic times, the motion is strongly correlated on short time scales. This is reflected in the peaks in the Fourier transforms of classical autocorrelation functions seen in a number of systems [1–4]. The peaks occur close to frequencies of dominant periodic motions, and their physical interpretation is that of resonances: the system preferentially exercises this motion, but due to its instability it cannot do so forever. For the axiom A systems a theory of such resonances has been developed by Ruelle [5–7], Pollicott [8] and others.

These resonances, as well as many other quantities of physical interest in dynamical systems, such as dimensions, Lyapunov exponents and entropies, can be expressed as averages over chaotic sets (repellers, 'strange' attractors, hyperbolic regions of the constant energy manifolds of classical Hamiltonian dynamics, the Julia sets of the holomorphic dynamics, and so on). An effective way to compute such averages is to rewrite them as averages over periodic orbits and then use the cycle expansions [9, 10] of the corresponding zeta functions [11, 12]. With this purpose in mind, we present here a zeta function (9) associated with smooth dynamical flows. As we have shown in [13], this zeta function is an efficient tool for accurate computation of classical resonances.

Consider a dynamical system described by d first-order differential equations $\dot{x} = \boldsymbol{F}(\boldsymbol{x})$. The corresponding flow $\boldsymbol{f}^t$ maps x_0 into $\boldsymbol{x} = \boldsymbol{f}^t(x_0)$ in time t. The evolution of a density $\rho(\boldsymbol{x})$ in this d-dimensional space is given by

$$\rho_t(\boldsymbol{x}) = \int \mathrm{d}y\, \rho(y)\delta(\boldsymbol{x} - \boldsymbol{f}^t(y)). \tag{1}$$

The kernel $\mathscr{L}^t = \delta(\boldsymbol{x} - \boldsymbol{f}^t(y))$ is the evolution operator (also known as the transfer, the Perron–Frobenius or the Ruelle–Araki operator in a variety of contexts). $\mathscr{L}^t$ is a linear operator, with eigenvalues $\mathrm{e}^{-\gamma_0 t}, \mathrm{e}^{-\gamma_1 t}, \ldots$. The physical significance of these eigenvalues is perhaps most easily grasped by considering the simplest application of zeta functions, evaluation of repeller escape rates. Given a finite enclosure V, the fraction of the initial volume that has not escaped by time t is given by [14]

$$\Gamma(t)=\int_V \mathrm{d}x\,\mathrm{d}y\,\delta(y-f^t(x))\left(\int_V \mathrm{d}x\right)^{-1}. \tag{2}$$

If the flow is unstable and locally expanding, this fraction is expected to decay exponentially, as $\Gamma(t)\approx \mathrm{e}^{-t\gamma}$. The escape rate is the asymptotic, $t\to\infty$ value of the decay exponent γ. The leading eigenvalue of $\mathscr{L}^t$ dominates the integral (2) and determines the escape rate, while the non-leading eigenvalues (possibly complex eigenvalue pairs, but with real parts less than the leading eigenvalue) describe correlations within the flow on the repeller [4, 6]. We shall evaluate here the spectrum of $\mathscr{L}^t$ by computing the trace

$$\operatorname{tr}\mathscr{L}^t=\int_V \mathrm{d}x\,\delta(x-f^t(x)). \tag{3}$$

For rigorous treatments (which require smearing out of the delta functions) we refer the reader to mathematical literature [15-18]; a good discussion of the difficulties inherent in the spectral theory of Perron-Frobenius operators is given in Dörfle [19].

According to the theorems of Ruelle and Pollicott for the axiom A systems and numerical evidence for a number of other systems, the spectrum of purely chaotic system consists of a series of resonances γ_α of multiplicity m_α,

$$\operatorname{tr}\mathscr{L}^t=\sum_{\alpha=0}^{\infty} m_\alpha\,\mathrm{e}^{-\gamma_\alpha t}. \tag{4}$$

The γ_α's are the positions of the resonances seen in the Fourier transforms of correlation functions, $\gamma_\alpha^{-1}=T_\alpha+\mathrm{i}\tau_\alpha$.

To evaluate the contribution of a prime periodic orbit p of period T_p to the trace, we go to a coordinate system with a longitudinal coordinate $\mathrm{d}x_\parallel$ along the direction of the flow $F(x(t))$, and $(d-1)$ transverse coordinates $x_\perp$ along a set of transverse directions $\hat{u}_1(t),\ldots,\hat{u}_{d-1}(t)$:

$$(\operatorname{tr}\mathscr{L}^t)_{|p}=\int_{V_p}\mathrm{d}x_\perp\,\mathrm{d}x_\parallel\,\delta_\perp(x-f^t(x))\delta_\parallel(x-f^t(x)). \tag{5}$$

Integration is restricted to a tube V_p in the neighbourhood of the cycle. The integral picks up a contribution whenever a trajectory returns to its starting point, i.e. multiple traversals of prime periodic orbits also have to be taken into account.

Evaluation of the transverse delta functions requires the linearization of the periodic flow in a plane perpendicular to the orbit,

$$\int_{V_p}\mathrm{d}x_\perp\,\delta_\perp(x-f^{rT_p}(x))=\frac{1}{|\det(\mathbf{1}-J_p^r)|} \tag{6}$$

where J_p is the $(d-1)\times(d-1)$ stability matrix, $\hat{u}(t+T_p)=J_p\hat{u}(t)$, evaluated on the p-cycle. Its eigenvalues $\Lambda_{p,1},\Lambda_{p,2},\ldots,\Lambda_{p,d-1}$ are independent of the position along the orbit and the choice of transverse coordinates. A geometrical interpretation of (6) is that after the rth return to a surface of section, the initial tube V_p has been stretched and squeezed along the stability matrix eigendirections perpendicular to the flow, with only the overlap with the initial volume given by $1/|\det(\mathbf{1}-J_p^r)|$.

Let $v=|\boldsymbol{F}(\boldsymbol{x})|$ be the velocity along the orbit and change variables $\mathrm{d}x_\parallel = v\,\mathrm{d}t'$. Whenever the time t' is a multiple of the period, the longitudinal delta function contributes a term $1/v$, cancelling the corresponding factor v from the change of variables, and the integral along the trajectory yields a factor T_p:

$$\int_{V_p} \mathrm{d}x_\parallel\, \delta_\parallel(\boldsymbol{x}-\boldsymbol{f}^t(\boldsymbol{x})) = \sum_{r=1}^{\infty} \delta(t-rT_p) \int_p \mathrm{d}t' = T_p \sum_{r=1}^{\infty} \delta(t-rT_p). \tag{7}$$

In the above, we have tacitly assumed hyperbolicity, i.e. that all eigenvalues are bounded away from unity, as the integral cannot be carried out if there are marginal eigenvalues $|\Lambda|=1$. A familiar example are families of periodic orbits in integrable Hamiltonian systems, where a single continuous coordinate distinguishes the orbits within a family. This coordinate has to be dealt with in a manner similar to the above integration along the orbit.

Substituting (6), (7) into (3), we obtain an expression for $\operatorname{tr}\mathscr{L}^t$ as a sum over all prime cycles p

$$\begin{aligned}\operatorname{tr}\mathscr{L}^t &= \sum_p T_p \sum_{r=1}^{\infty} \frac{\delta(t-rT_p)}{|\det(\mathbf{1}-\boldsymbol{J}_p^r)|} \\ &= \frac{\mathrm{i}}{2\pi}\int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kt} \frac{\partial}{\partial k}\sum_p \sum_{r=1}^{\infty} \frac{1}{r}\frac{\mathrm{e}^{-\mathrm{i}kT_p r}}{|\det(\mathbf{1}-\boldsymbol{J}_p^r)|}\end{aligned} \tag{8}$$

where in the last step we have replaced the δ function by its Fourier representation. After a rotation $k\to \mathrm{i}s$, it is easy to check that the above sums are the logarithmic derivative of the zeta function

$$Z(s)=\exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r}\frac{\mathrm{e}^{sT_p r}}{|\det(\mathbf{1}-\boldsymbol{J}_p^r)|}\right) \tag{9}$$

so putting (4) and (8) together we can identify

$$\sum_{\alpha=0}^{\infty} m_\alpha\, \mathrm{e}^{-\gamma_\alpha t} = \frac{\mathrm{i}}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\, \mathrm{e}^{-st} Z'(s)/Z(s). \tag{10}$$

If $Z(s)$ is an entire function, the integration contour can be deformed to encircle the poles of $Z'(s)/Z(s)$ and pick up a contribution $m_\alpha\, \mathrm{e}^{-\gamma_\alpha t}$ from each pole $s=\gamma_\alpha$ of multiplicity m_α. Hence the zeros of $Z(s)$ determine the spectrum $\gamma_0, \gamma_1, \ldots$, as asserted in the introduction, and the determination of resonances is now reduced to a computation of zeros of the zeta function (9).

In order to rewrite $Z(s)$ as an infinite product over periodic orbits, we note that the r sum (9) is close in form to expansion of a logarithm. To bring (9) to such form, we factorize the denominator determinants into products of expanding and contracting eigenvalues. For concreteness, consider a three-dimensional flow with one expanding eigenvalue Λ (of absolute value >1) and one contracting eigenvalue λ, with $|\lambda|<1$. (An example are conservative two degree of freedom systems restricted to the energy shell; extension to higher dimensions is straightforward [9, 10].) Then the determinant in (9) may be expanded as follows:

$$|\det(\mathbf{1}-\boldsymbol{J}_p^r)|^{-1} = |(1-\Lambda_p^r)(1-\lambda_p^r)|^{-1} = |\Lambda_p|^{-r} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Lambda_p^{-jr}\lambda_p^{kr}. \tag{11}$$

With this we can rewrite the exponent in (9) as

$$\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{sT_p r}}{|\det(\mathbf{1}-\boldsymbol{J}_p^r)|} = \sum_p \sum_{j,k} \sum_{r=1}^{\infty} \frac{1}{r} (e^{sT_p} |\Lambda_p|^{-1} \Lambda_p^{-j} \lambda_p^k)^r$$
$$= \sum_p \sum_{j,k} \log(1 - e^{sT_p} |\Lambda_p|^{-1} \Lambda_p^{-j} \lambda_p^k) \tag{12}$$

and bring $Z(s)$ to a 'Selberg product' form [20]:

$$Z(s) = \prod_p \prod_{j,k=0}^{\infty} \left(1 - \frac{e^{sT_p} z^{n_p}}{|\Lambda_p|} \frac{\lambda_p^k}{\Lambda_p^j}\right). \tag{13}$$

Readers familiar with the Fredholm determinants $Z(z) = \det(1 - z\mathscr{L})$ for maps [11, 12] will note that the continuous time $Z(s)$ is a natural generalization of $Z(z)$ for the discrete time case, with the p-cycle weight $z^{n_p}/|\det(\mathbf{1} - \boldsymbol{J}_p)|$ replaced by the $z \to e^s$ continuous time weight $e^{sT_p}/|\det(\mathbf{1} - \boldsymbol{J}_p)|$. Indeed, (9) is essentially the zeta function for flows studied by Ruelle [5, 15], Pollicott [8], Fried [21] and others. The main difference between our result and the zeta functions of the above authors is that they project the flow onto the unstable manifold, and keep only the expanding eigenvalues in the expansion [12]. This amounts to dropping the product over k in (13) and the exponent $k+1$ in (14); while this reduction does not affect the leading eigenvalue, the resulting spectrum is not the physically interesting correlation spectrum (4).

Evaluation of (13) by means of cycle expansions is discussed in detail elsewhere [9, 10]. Briefly, we find it convenient to introduce a book-keeping parameter z^{n_p} in (13), with n_p the topological cycle length, i.e. the number of times the p-cycle crosses a Poincaré surface of section. One then expands the infinite product, arranges it in a power series in z, and after truncating the series sets $z = 1$ in the final computation. One example of such calculation has been presented in [13], and others are in preparation.

Equations (9), (13) are the main result of this communication. We conclude with several remarks.

The mathematical literature derivations of $Z(s)$ are based on the flow suspension approach [22], in which a flow is replaced by a Poincaré map with appropriate measure. Here we have presented an alternative derivation, replacing the generating sum manipulations of $\operatorname{tr} \mathscr{L}^n$ used in the discrete time $Z(z)$ derivations by integral transforms of $\operatorname{tr} \mathscr{L}^t$, in order to emphasize the similarity to the derivation of the *quantum zeta function* [23, 24].

Periodic orbit expansions figure prominently in the semiclassical description of chaotic systems, where an expression for the quantum energy spectrum in terms of cycles of the classical flow has been given by Gutzwiller [23, 24]. For 2D Hamiltonian systems $\lambda = 1/\Lambda$ and the classical $Z(s)$ may be written as

$$Z(s) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{sT_p}}{|\Lambda_p| \Lambda_p^k}\right)^{k+1} \tag{14}$$

whereas the quantum [24, 25] equivalent reads

$$Z(E) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{-iS_p(E)/\hbar + \nu_p}}{\sqrt{|\Lambda_p|}\, \Lambda_p^k}\right) \tag{15}$$

with S_p the classical action, and ν_p the Maslov index. Both zeta functions relate the spectrum of an evolution operator to periodic orbits of a flow; in the classical context, the operator is a Dirac delta function, and the above formulae for $Z(s)$ are exact, while in the quantum context the operator traces have so far been evaluated only in a stationary phase approximation. Remarkably enough, for spaces of constant negative curvature, also (15) is exact; in this case the length, the period and the instability of a geodesic are all proportional to each other, $l_p \propto T_p \propto \ln \Lambda_p$, and (15) is the Selberg zeta function [20].

The differences between the classical and quantum zetas reflect the fact that in classical mechanics probabilities are added (whence the full determinant in the denominators in (9)), whereas in quantum mechanics amplitudes are superimposed, hence the square root of the eigenvalue in [15].

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